



# Effect of initial twist on wave characteristics in a prestressed fluid-filled elastic thin tube

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*In this work the effect of initial twist on wave characteristics in a prestressed elastic thin tube filled with a viscous fluid is studied. Initially the tube is subjected to a large static transmural pressure  $P_i$ , an axial stretch  $\lambda_z$ , and a twisting moment  $M$ . Assuming that in the course of fluid flow a small incremental axially symmetric disturbance is added on this field the equations governing the incremental motion are obtained both for elastic tube and viscous fluid. Seeking a harmonic wave type of solution to the field equations and using the boundary conditions the dispersion relation is obtained for long-wave approximation. Considering the difficulties of analytical treatment of the dispersion equation a numerical analysis is presented and the results are discussed.*

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## 1. Introduction

Propagation of harmonic waves in initially stressed (or unstressed) circularly cylindrical tubes filled with a viscous (or inviscid) fluid is a problem of interest since the time of Thomas Young, who first calculated the pulse wave speed in human arteries. The current literature on the subject is so rich that it is almost impossible to cite all those contributions here. The historical evolution of the subject may be found in the books by McDonald<sup>1</sup> and Fung<sup>2</sup> and in the survey papers by Lambossy<sup>3</sup> and Skalak.<sup>4</sup> Significant contributions on symmetrical wave motions in elastic tubes filled with a viscous fluid have been made by Witzig,<sup>5</sup> Morgan and Kiely,<sup>6</sup> Womersley,<sup>7</sup> Atabek and Lew,<sup>8</sup> and more recently by Rachev.<sup>9</sup> The propagation of torsional waves in fluid-filled thin tubes has been studied by Maxwell and Anliker,<sup>10</sup> Moodie and Barclay,<sup>11</sup> Demiray,<sup>12</sup> and Demiray and Ercengiz.<sup>13</sup> In the majority of these works, either the effects of initial stresses have been neglected or taken into consideration in an *ad hoc* manner. Moreover the elastic or viscoelastic coefficients of the incremental stress have been treated as some material constants. In

reality these coefficients are some functions of the initial deformation, and hence the value of them changes with deformation.

As is well known, for a healthy young human being the diastolic pressure is 80 mm Hg, the systolic pressure is 120 mm Hg, and the mean pressure is around 100 mm Hg. Furthermore, under physiological conditions, large blood vessels are subjected to an axial stretch  $\lambda_z$ , which is about 1.5. Besides these effects, due to improper functioning of heart valves, the end pressure applied by the left ventricle may not be uniform through the circumference of the arterial wall. In such cases it is possible to have a twist in the arterial wall. This is equivalent to stating that the arteries are initially subjected to a large static deformation. In the course of blood flow, upon this initial static field a pressure deviation  $\pm 20$  mm Hg is applied by the left ventricle. Considering the stiffening properties of soft biological tissues with stress the dynamic displacements resulting from this pressure deviation and fluid motion may be assumed to be small as compared to the initial static field. Therefore the theory of small deformation superimposed on initial large static deformation may be employed in analyzing the harmonic wave propagation in large blood vessels.

In the present work, utilizing the theory of small displacements superimposed on large initial static deformation, the propagation of harmonic waves in an initially inflated, axially stretched and twisted elastic thin tube filled with a viscous fluid is studied. Considering the physiological conditions the stress distribution resulting

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from uniform inner pressure, axial stretch, and circumferential twist is obtained. Superimposing a small but dynamic displacement field upon this static field the governing incremental equations of motion are obtained in the cylindrical polar coordinates. Seeking a harmonic wave type of solution to the field equations of the fluid and the tube, and employing the boundary conditions, the dispersion relation is obtained for the long-wave approximation. Due to the difficulty of the analysis of the dispersion equation by analytical means, a numerical technique is employed, and the variations of propagation speeds and the transmission coefficients are evaluated with a Womersley parameter, the stretch ratios and the twist parameter and the results are depicted in graphical forms. It is shown that, although the incremental motion is axially symmetric, due to the initial twist of the tube material the torsional wave is coupled with two other waves propagating in such a composite medium.

## 2. Basic equations

Due to interactions of blood with its container the pulsatile motion of blood, resulting from the periodic motion of the heart, leads to the wave phenomena in arteries. Therefore the governing equations and the boundary conditions should include these interactions.

### 2.1 Equations of fluid

Blood is known to be an incompressible non-Newtonian fluid. However, in the course of blood flow in arteries, small velocity and pressure increments are added on the existing uniform initial transmural pressure field  $P_i$ . Therefore the incremental behavior of the blood may be treated as Newtonian. Thus the equations of symmetrical motion in the cylindrical polar coordinates may be expressed as follows

$$\begin{aligned} -\frac{\partial \bar{p}}{\partial r} + \mu_v \left( \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right) - \bar{p} \frac{\partial \bar{u}}{\partial t} &= 0, \\ \mu_v \left( \frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r^2} + \frac{\partial^2 \bar{v}}{\partial z^2} \right) - \bar{p} \frac{\partial \bar{v}}{\partial t} &= 0, \\ -\frac{\partial \bar{p}}{\partial z} + \mu_v \left( \frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}}{\partial r} + \frac{\partial^2 \bar{w}}{\partial z^2} \right) - \bar{p} \frac{\partial \bar{w}}{\partial t} &= 0, \end{aligned} \quad (1)$$

and the incompressibility condition as

$$\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r} + \frac{\partial \bar{w}}{\partial z} = 0, \quad (2)$$

where  $\bar{p}$  is the mass density,  $\bar{p}$  is the pressure increment,  $\mu_v$  is the viscosity,  $\bar{u}$ ,  $\bar{v}$ , and  $\bar{w}$  are, respectively, the velocity components of the fluid body in the radial, circumferential, and axial directions. The stress components

that we need in using the boundary conditions are given by

$$\begin{aligned} \bar{\sigma}_{rr} &= -\bar{p} + 2\mu_v \frac{\partial \bar{u}}{\partial r}, \quad \bar{\sigma}_{r\theta} = \mu_v \left( \frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r} \right) \\ \bar{\sigma}_{rz} &= \mu_v \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r} \right) \end{aligned} \quad (3)$$

### 2.2 Equations of solid body

The arterial wall material is known to be incompressible, anisotropic, and viscoelastic (cf. Fung<sup>14</sup> and Cox<sup>15</sup>). However, for its mathematical simplicity in nonlinear analysis, the arterial wall material will be assumed to be incompressible, homogeneous, isotropic, and elastic. In the present work we shall adopt the constitutive relations proposed by Demiray<sup>16</sup> for such a soft biological tissue, as,

$$t^{kl} = \pi g^{kl} + \mu \exp[\gamma(I_1 - 3)]c^{kl} \quad (4)$$

where  $c^{kl} = G^{KL}F_K^kF_L^l$  is the Finger deformation tensor,  $F_K^k \equiv \partial x^k / \partial X^K$  is the deformation gradient,  $G^{KL}$  and  $g^{kl}$  are, respectively, the reciprocal metric tensors of the material and spatial frames,  $I_1 = c_1^1 + c_2^2 + c_3^3$  is the first invariant of Finger deformation tensor,  $t^{kl}$  is the Cauchy stress tensor,  $\pi$  is the hydrostatic pressure to be determined from the field equations and the boundary conditions, and  $\mu$  and  $\gamma$  are two material constants to be determined from experimental measurements.

Now let us consider a circularly cylindrical thin tube made of such an incompressible and isotropic-elastic material subjected to a large transmural pressure  $P_i$ , axial stretch  $\lambda_z$  (or axial force  $N$ ), and a torque  $M$  in the circumferential direction. Upon applications of such a static loading the following type of deformation, in the cylindrical polar coordinates, will be developed in the tube

$$r = (R^2/\lambda_z + A)^{1/2}, \quad \theta = \Theta + KZ, \quad z = \lambda_z Z \quad (5)$$

where  $(R, \Theta, Z)$  and  $(r, \theta, z)$  are the cylindrical polar coordinates of a material point before and after deformation,  $K$  is the twist angle, and  $A$  is an integration constant to be determined from the boundary conditions. Here we should point out that such a twist of the artery may result from the nonsymmetrical end condition created by the left ventricle.

Introducing equation (5) into (4) the nonvanishing stress components read

$$\begin{aligned} t_{rr}^0 &= \pi^0 + \frac{\mu}{\lambda_\theta^2 \lambda_z^2} F(\lambda_\theta, \lambda_z), \\ t_{\theta\theta}^0 &= \pi^0 + \mu(\lambda_\theta^2 + K^2 r^2) F(\lambda_\theta, \lambda_z) \\ t_{zz}^0 &= \pi^0 + \mu \lambda_z^2 F(\lambda_\theta, \lambda_z), \quad t_{\theta z}^0 = \mu \lambda_z K r F(\lambda_\theta, \lambda_z) \\ F(\lambda_\theta, \lambda_z) &\equiv \exp \left[ \gamma \left( \lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2 \lambda_z^2} + K^2 r^2 - 3 \right) \right], \\ \lambda_\theta &\equiv \frac{r}{R} \end{aligned} \quad (6)$$

These stress components must satisfy the Cauchy's equations of equilibrium, which read in the cylindrical polar coordinates,

$$\frac{\partial t_{rr}^0}{\partial r} + \frac{1}{r}(t_{rr}^0 - t_{\theta\theta}^0) = 0. \quad (7)$$

Introducing equation (6) into (7), integrating the result with respect to  $r$ , and using the boundary condition on the outer surface, i.e.,  $t_{rr}(r_o) = 0$ , we obtain

$$t_{rr}^0 = \mu \int_{r_o}^r F(\lambda_\theta \lambda_z) \left( \lambda_\theta^2 - \frac{1}{\lambda_\theta^2 \lambda_z^2} \right) \frac{dr}{r} + \mu K^2 \int_{r_o}^r F(\lambda_\theta, \lambda_z) r dr, \quad (8)$$

where  $r_o$  denotes the deformed outer radius of the tube. Since the artery is subjected to the inner pressure  $P_i$ , from equation (8), the pressure deformation relation reads

$$P_i = \mu \int_{r_i}^{r_o} F(\lambda_\theta \lambda_z) \left( \lambda_\theta^2 - \frac{1}{\lambda_\theta^2 \lambda_z^2} \right) \frac{dr}{r} + \mu K^2 \int_{r_i}^{r_o} F(\lambda_\theta, \lambda_z) r dr. \quad (9)$$

In this work we are interested in thin tubes, for which case the radial stress component  $t_{rr}$  may be taken to be zero. From this assumption and equation (6) the hydrostatic pressure term becomes

$$\pi^0 = -\frac{\mu}{\bar{\lambda}_\theta^2 \lambda_z^2} F(\bar{\lambda}_\theta, \lambda_z). \quad (10)$$

Thus the other stress components take the following form

$$\begin{aligned} t_{\theta\theta}^0 &= \mu F(\bar{\lambda}_\theta, \lambda_z) \left[ \left( \bar{\lambda}_\theta^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) + K^2 \bar{r}^2 \right] \\ t_{zz}^0 &= \mu F(\bar{\lambda}_\theta, \lambda_z) \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \\ t_{\theta z}^0 &= \mu \lambda_z K \bar{r} F(\bar{\lambda}_\theta, \lambda_z), \quad \bar{\lambda}_\theta \equiv \frac{\bar{r}}{R}, \end{aligned} \quad (11)$$

where  $\bar{R}$  and  $\bar{r}$  are, respectively, the midradius of the tube before and after deformation. Similarly, for thin tubes, the inner pressure may be approximated by

$$P_i = \mu F(\bar{\lambda}_\theta, \lambda_z) \left[ \left( \bar{\lambda}_\theta^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \frac{h}{\bar{r}} + K^2 \bar{r} h \right] \quad (12)$$

where  $h$  is the deformed thickness of the tube, related to the undeformed thickness  $H$  through

$$h \equiv \frac{H}{\bar{\lambda}_\theta \lambda_z}. \quad (13)$$

The torque  $M$  applied to the tube may be expressed as follows

$$M = 2\pi \mu K \bar{r}^3 h F(\bar{\lambda}_\theta, \lambda_z). \quad (14)$$

Now, upon this initial deformation, we shall superimpose a small displacement field  $\mathbf{u}(z, t)$ , or in components form,  $u_1 = u(z, t)$ ,  $u_2 = v(z, t)$ ,  $u_3 = w(z, t)$ ; where  $u$ ,  $v$ , and  $w$  are the radial, circumferential, and axial displacement components in the cylindrical polar coordinates. A material point located at  $(\bar{r}, \theta, z)$  on the midsurface will move to a new position  $(\bar{r} + u, \theta + v/\bar{r}, z + w)$  after superimposing this small displacement field. Thus the position vector after final deformation may be expressed by

$$\mathbf{r} = (\bar{r} + u)\mathbf{e}_r + v\mathbf{e}_\theta + (z + w)\mathbf{e}_z. \quad (15)$$

The vector tangent to the generator of the cylindrical tube before this final deformation transforms into another vector defined by

$$\mathbf{T}_1 = \frac{\partial \mathbf{r}}{\partial z} = u_{,z}\mathbf{e}_r + v_{,z}\mathbf{e}_\theta + (1 + w_{,z})\mathbf{e}_z. \quad (16)$$

Here for the sake of brevity we have defined  $(\cdot)_{,z} \equiv \partial(\cdot)/\partial z$ . For small incremental deformation the length of the vector  $\mathbf{T}_1$  may be approximated by

$$|\mathbf{T}_1| = \Lambda_1 \cong (1 + w_{,z}). \quad (17)$$

Thus the unit vector  $\mathbf{t}_1$  along  $\mathbf{T}_1$  may be given by

$$\mathbf{t}_1 = \frac{\mathbf{T}_1}{\Lambda_1} \cong u_{,z}\mathbf{e}_r + v_{,z}\mathbf{e}_\theta + \mathbf{e}_z. \quad (18)$$

The vector  $\bar{\mathbf{r}}\mathbf{e}_\theta$  in the circumferential direction before this small deformation transforms into  $\mathbf{T}_2$  after the deformation and may be given by

$$\mathbf{T}_2 = \frac{\partial \mathbf{r}}{\partial \theta} = -v\mathbf{e}_r + (\bar{r} + u)\mathbf{e}_\theta. \quad (19)$$

The length of this vector may be approximated by

$$|\mathbf{T}_2| = \Lambda_2 \cong \bar{r} + u. \quad (20)$$

Hence the unit vector  $\mathbf{t}_2$  along  $\mathbf{T}_2$  is given by

$$\mathbf{t}_2 = \frac{\mathbf{T}_2}{\Lambda_2} \cong -\frac{v}{\bar{r}}\mathbf{e}_r + \mathbf{e}_\theta. \quad (21)$$

The external unit normal vector  $\mathbf{n}$  to this deformed midsurface is given

$$\mathbf{n} = \mathbf{t}_2 \times \mathbf{t}_1 = \mathbf{e}_r + \frac{v}{\bar{r}}\mathbf{e}_\theta - u_{,z}\mathbf{e}_z. \quad (22)$$

Let the external force acting per unit area of the tube be represented by

$$\mathbf{P} = P_1\mathbf{t}_1 + P_2\mathbf{t}_2 + P_n\mathbf{n}, \quad (23)$$

where  $P_1$ ,  $P_2$ , and  $P_n$  are the components, in respective directions, of the external force resulting from fluid-tube interactions. Then the total force acting on the small tube element (see Figure 1) may be given by

$$\mathbf{F} = \left\{ \frac{\partial}{\partial z} [(\bar{r} + u)(N_1 \mathbf{t}_1 + N_{12} \mathbf{t}_2)] + \frac{\partial}{\partial \theta} [(1 + w, z)(N_2 \mathbf{t}_2 + N_{21} \mathbf{t}_1)] + (P_1 \mathbf{t}_1 + P_2 \mathbf{t}_2 + P_n \mathbf{t}_n)(\bar{r} + u + \bar{r}w, z) \right\} d\theta dz, \quad (24)$$

where  $N_1$  and  $N_2$  are the resultant forces in the meridional and circumferential directions, and  $N_{21}$  and  $N_{12}$  are the resultant shear forces acting in the meridional and circumferential directions, respectively.

In this work, since the superimposed deformation is assumed to be small, equation (24) may be linearized. For that purpose we set

$$\begin{aligned} N_1 &= N_z^0 + \Sigma_1, & N_2 &= N_{\theta}^0 + \Sigma_2, & N_{21} &= N_{\theta z}^0 + \Sigma_{12} \\ N_{12} &= N_{z\theta}^0 + \Sigma_{21}, & P_1 &= 0 + \bar{P}_1, \\ P_2 &= 0 + \bar{P}_2, & P_n &= P_i + \bar{P}_n \end{aligned} \quad (25)$$

where  $N_z^0, \dots, N_{\theta z}^0$  are the initial membrane forces and  $\Sigma_1, \dots, \bar{P}_n$  are the increments of the corresponding quantities resulting from the incremental deformation. Introducing equation (25) into (24) and neglecting the higher order terms in the incremental quantities one obtains

$$\begin{aligned} \mathbf{F} = & \left\{ \left[ \bar{r} N_z^0 \frac{\partial^2 u}{\partial z^2} - N_{z\theta}^0 \frac{\partial v}{\partial z} - N_{\theta}^0 - N_{\theta}^0 \frac{\partial w}{\partial z} - N_{\theta z}^0 \frac{\partial v}{\partial z} \right. \right. \\ & \left. \left. - \Sigma_2 + \bar{r} P_i + \bar{r} \bar{P}_n + P_i \left( u + \bar{r} \frac{\partial w}{\partial z} \right) \right] \mathbf{e}_r \right. \\ & + \left[ \bar{r} N_z^0 \frac{\partial^2 v}{\partial z^2} + \bar{r} \frac{\partial \Sigma_{12}}{\partial z} + 2 N_{z\theta}^0 \frac{\partial u}{\partial z} + \bar{r} \bar{P}_2 \right] \mathbf{e}_{\theta} \\ & \left. + \left[ (N_z^0 - N_{\theta}^0) \frac{\partial u}{\partial z} + \bar{r} \frac{\partial \Sigma_1}{\partial z} + \bar{r} \bar{P}_1 \right] \mathbf{e}_z \right\} d\theta dz. \end{aligned} \quad (26)$$

This force should be equal to the mass times the acceleration of this small element and, in linearized form, it may be given by

$$\rho \bar{r} \left( \frac{\partial^2 u}{\partial t^2} \mathbf{e}_r + \frac{\partial^2 v}{\partial t^2} \mathbf{e}_{\theta} + \frac{\partial^2 w}{\partial t^2} \mathbf{e}_z \right), \quad (27)$$

where  $\rho$  is the mass density of the tube material. Equating equations (26) to (27), the equations of motion, in

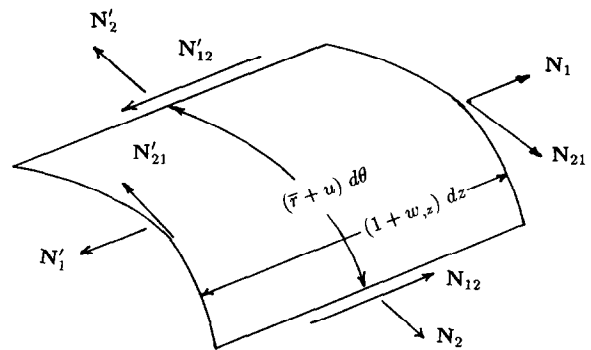


Figure 1. Membrane forces acting on a small tube element.

component form take the following form

$$N_z^0 \frac{\partial^2 u}{\partial z^2} + \frac{N_{\theta}^0}{\bar{r}^2} u - \frac{2 N_{z\theta}^0}{\bar{r}} \frac{\partial v}{\partial z} - \frac{\Sigma_2}{\bar{r}} + \bar{P}_n = \rho h \frac{\partial^2 u}{\partial t^2} \quad (28)$$

$$N_z^0 \frac{\partial^2 v}{\partial z^2} + \frac{2 N_{\theta z}^0}{\bar{r}} \frac{\partial u}{\partial z} + \frac{\partial \Sigma_{12}}{\partial z} + \bar{P}_2 = \rho h \frac{\partial^2 v}{\partial t^2} \quad (29)$$

$$(N_z^0 - N_{\theta}^0) \frac{\partial u}{\partial z} + \frac{\partial \Sigma_1}{\partial z} + \bar{P}_n = \rho h \frac{\partial^2 w}{\partial t^2}. \quad (30)$$

In obtaining these equations we have utilized the following relation, which is known as the Laplace Law,

$$N_{\theta}^0 = \bar{r} P_i. \quad (31)$$

In order to complete the governing field equations one must know the expressions of the forces applied by the viscous fluid. As was assumed before the fluid is initially subjected to a uniform pressure  $P_i$ . Upon application of the incremental field the total stress tensor of the fluid may be expressed by

$$\sigma_{kl} = -P_i \delta_{kl} + \bar{\sigma}_{kl}, \quad (32)$$

where  $\bar{\sigma}_{kl}$  is the incremental stress tensor of the fluid and given by

$$\bar{\sigma}_{kl} = -\bar{p} \delta_{kl} + 2\mu_v \bar{d}_{kl}; \quad \bar{d}_{kl} \equiv \frac{1}{2} (\bar{v}_{k,l} + \bar{v}_{l,k}), \quad (33)$$

where  $\bar{p}$  is the increment in fluid pressure and the indices following a comma are used to denote the partial differentiation with respect to that coordinate. The components of the reaction force on the surface of the tube are defined by

$$\begin{aligned} P_1 &= -\mathbf{t}_1 \sigma \mathbf{n}|_{r=\bar{r}}, & P_2 &= -\mathbf{t}_2 \sigma \mathbf{n}|_{r=\bar{r}}, \\ P_n &= -\mathbf{n} \sigma \mathbf{n}|_{r=\bar{r}}. \end{aligned} \quad (34)$$

Noticing that  $\bar{\sigma}_{kl}$  is of the same order of the other incremental quantities and using equation (22) the trac-

tion components become

$$\begin{aligned} P_1 = \bar{P}_1 = -\bar{\sigma}_{rz}|_{r=\bar{r}}, \quad P_2 = \bar{P}_2 = -\bar{\sigma}_{r\theta}|_{r=\bar{r}} \\ P_n = \bar{P}_n = -\bar{\sigma}_{rr}|_{r=\bar{r}}. \end{aligned} \quad (35)$$

Introducing equation (35) into equations (28)–(30) the following equations are obtained

$$N_z^0 \frac{\partial^2 u}{\partial z^2} + \frac{N_\theta^0}{\bar{r}^2} u - \frac{2N_{\theta z}^0}{\bar{r}} \frac{\partial v}{\partial z} - \frac{\Sigma_2}{\bar{r}} - \bar{\sigma}_{rr}|_{r=\bar{r}} = \rho h \frac{\partial^2 u}{\partial t^2}, \quad (36)$$

$$N_z^0 \frac{\partial^2 v}{\partial z^2} + \frac{2N_{\theta z}^0}{\bar{r}} \frac{\partial u}{\partial z} + \frac{\partial \Sigma_{12}}{\partial z} - \bar{\sigma}_{r\theta}|_{r=\bar{r}} = \rho h \frac{\partial^2 v}{\partial t^2}, \quad (37)$$

$$\left( \frac{N_z^0 - N_\theta^0}{\bar{r}} \right) \frac{\partial u}{\partial z} + \frac{\partial \Sigma_1}{\partial z} - \bar{\sigma}_{rz}|_{r=\bar{r}} = \rho h \frac{\partial^2 w}{\partial t^2}. \quad (38)$$

In these equations the explicit expressions of the incremental membrane forces are unknown. In order to complete the field equations of the membrane the expressions of the incremental force resultants  $\Sigma_1$ ,  $\Sigma_2$ , and  $\Sigma_{12} = \Sigma_{21}$  must be given. Let  $t_{kl}$  be the stress tensor defined in the final configuration. By definition the stress resultants  $N_1$ ,  $N_2$ , and  $N_{12}$  may be given by

$$N_1 = h' \sigma_1, \quad N_2 = h' \sigma_2, \quad N_{12} = h' \sigma_{12}, \quad (39)$$

where  $h'$  is the thickness of the tube in the final configuration and  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_{12}$  are defined by

$$\sigma_1 = t_{kl} t_k^{(1)} t_l^{(1)}, \quad \sigma_2 = t_{kl} t_k^{(2)} t_l^{(2)}, \quad \sigma_{12} = t_{kl} t_k^{(1)} t_l^{(2)}, \quad (40)$$

where  $t_k^{(\alpha)}$  are the components of the unit vectors  $\mathbf{t}_\alpha$  in the cylindrical polar coordinates. Introducing the expressions of  $\mathbf{t}_1$  and  $\mathbf{t}_2$  into equation (40) and keeping only the linear terms in the incremental quantities we have

$$\sigma_1 = t_{zz} + 2t_{\theta z} \frac{\partial v}{\partial z}, \quad \sigma_2 = t_{\theta\theta}, \quad \sigma_{12} = t_{z\theta} + t_{\theta\theta} \frac{\partial v}{\partial z}. \quad (41)$$

Here, for thin tubes, we have already used the approximations  $t_{rr} \cong 0$ ,  $t_{r\theta} \cong 0$ ,  $t_{rz} \cong 0$ .

Since the material under investigation is assumed to be incompressible we have

$$\bar{r} h d\theta dz = (\bar{r} + u)(1 + w_z) h' d\theta dz,$$

or, solving this equation for  $h'$  one gets

$$h' = \frac{h}{(1 + u/\bar{r})(1 + w_z)} \cong h \left( 1 - \frac{u}{\bar{r}} - w_z \right). \quad (42)$$

Introducing the incremental stress tensor  $\bar{t}_{kl}$  as  $t_{kl} = t_{kl}^0 + \bar{t}_{kl}$  in equation (39) and utilizing the approximations (41)

and (42) we have

$$\begin{aligned} \Sigma_1 &= h \bar{t}_{zz} + 2N_{\theta z}^0 \frac{\partial v}{\partial z} - N_z^0 \left( \frac{u}{\bar{r}} + \frac{\partial w}{\partial z} \right) \\ \Sigma_2 &= h \bar{t}_{\theta\theta} - N_\theta^0 \left( \frac{u}{\bar{r}} + \frac{\partial w}{\partial z} \right) \\ \Sigma_{12} &= h \bar{t}_{\theta z} + N_\theta^0 \frac{\partial v}{\partial z} - N_{\theta z}^0 \left( \frac{u}{\bar{r}} + \frac{\partial w}{\partial z} \right). \end{aligned} \quad (43)$$

To determine the explicit expressions of  $\Sigma_1$ ,  $\Sigma_2$ , and  $\Sigma_{12}$  we shall consult the theory of so-called "small displacements superimposed on large static deformation." The derivation of the governing differential equations and the constitutive relations for this type of deformation had been given in the books by Green and Zerna<sup>17</sup> and Eringen and Suhubi.<sup>18</sup> For the specific problem that we are studying here the physical components are given by

$$\begin{aligned} \Sigma_1 &= h(\alpha_{11}^0 e_{11} + \alpha_{12}^0 e_{22} + 2\alpha_{13}^0 e_{12}) \\ \Sigma_2 &= h(\alpha_{21}^0 e_{11} + \alpha_{22}^0 e_{22} + 2\alpha_{23}^0 e_{12}) \\ \Sigma_{12} &= h(\alpha_{31}^0 e_{11} + \alpha_{32}^0 e_{22} + 2\alpha_{33}^0 e_{12}), \end{aligned} \quad (44)$$

where the coefficients  $\alpha_{ij}^0$  ( $i, j = 1, 2, 3$ ) and the infinitesimal strain components  $e_{\alpha\beta}$  ( $\alpha, \beta = 1, 2$ ) are defined by

$$\begin{aligned} \alpha_{11}^0 &= \mu F \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} + \lambda_z^2 \right) + 2\gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right)^2 \right] \\ \alpha_{12}^0 &= \mu F \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} - \lambda_z^2 \right) + 2\gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right. \\ &\quad \times \left. \left( \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] \\ \alpha_{13}^0 &= 2\mu \lambda_z K \bar{r} F \left[ 1 + \gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] \\ \alpha_{21}^0 &= \mu F \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} - \bar{\lambda}_\theta^2 - K^2 \bar{r}^2 \right) \right. \\ &\quad \times \left. 2\gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \left( \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] \\ \alpha_{22}^0 &= \mu F \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} + \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 \right) \right. \\ &\quad \times \left. 2\gamma \left( \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right)^2 \right] \\ \alpha_{23}^0 &= 2\mu \lambda_z K \bar{r} F \left[ 1 + \gamma \left( \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] \\ \alpha_{31}^0 &= 2\gamma \mu \lambda_z K \bar{r} F \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \\ \alpha_{32}^0 &= 2\gamma \mu \lambda_z K \bar{r} \left( \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \end{aligned}$$

$$\alpha_{33}^0 = \mu F \left( \lambda_z^2 + \bar{\lambda}_\theta^2 + K^2 \bar{r}^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} + 2\gamma K^2 \bar{r}^2 \right), \quad (45)$$

and

$$e_{11} = \frac{\partial w}{\partial z}, \quad e_{22} = \frac{u}{\bar{r}}, \quad e_{12} = \frac{1}{2} \frac{\partial v}{\partial z}. \quad (46)$$

Introducing equation (44) into equations (36)–(38) we obtain

$$N_z^0 \frac{\partial^2 u}{\partial z^2} + \frac{(N_\theta^0 - h\alpha_{22}^0)}{\bar{r}^2} u - \frac{(2N_{\theta z}^0 + h\alpha_{23}^0)}{\bar{r}} \frac{\partial v}{\partial z} - \frac{h\alpha_{21}^0}{\bar{r}} \frac{\partial w}{\partial z} - \bar{\sigma}_{rr}|_{r=\bar{r}} = \rho h \frac{\partial^2 u}{\partial t^2}, \quad (47)$$

$$(N_z^0 + h\alpha_{33}^0) \frac{\partial^2 v}{\partial z^2} + \frac{(2N_{\theta z}^0 + h\alpha_{32}^0)}{\bar{r}} \frac{\partial u}{\partial z} + h\alpha_{31}^0 \frac{\partial^2 w}{\partial z^2} - \bar{\sigma}_{r\theta}|_{r=\bar{r}} = \rho h \frac{\partial^2 v}{\partial t^2}, \quad (48)$$

$$\frac{(N_z^0 - N_\theta^0 + h\alpha_{12}^0)}{\bar{r}} \frac{\partial u}{\partial z} + h\alpha_{11}^0 \frac{\partial^2 w}{\partial z^2} + h\alpha_{13}^0 \frac{\partial^2 v}{\partial z^2} - \bar{\sigma}_{rz}|_{r=\bar{r}} = \rho h \frac{\partial^2 w}{\partial t^2}. \quad (49)$$

The equations (1), (2), and (47)–(49) are to be solved under the following boundary conditions

$$\bar{u}(\bar{r}, z, t) = \frac{\partial u}{\partial t}, \quad \bar{v}(\bar{r}, z, t) = \frac{\partial v}{\partial t}, \quad \bar{w}(\bar{r}, z, t) = \frac{\partial w}{\partial t}. \quad (50)$$

Here we should point out that, although the motion of the fluid in the circumferential direction is not coupled with the motions in the radial and axial directions, due to the initial twist of the tube all the field quantities are coupled. However, from the standpoint of the boundary conditions in equation (50), the velocity components of the fluid are also coupled.

### 3. Solution to field equations

In this section we shall seek a harmonic wave type of solution to the differential equations given in equations (1), (2), and (47)–(49). For such motions the proper form of the field quantities should be given as follows

$$\{\bar{p}, \bar{u}, \bar{v}, \bar{w}\} = \{\bar{P}(r), \bar{U}(r), \bar{V}(r), \bar{W}(r)\} \times \exp[i(\omega t - kz)] \quad (51)$$

$$\{u, v, w\} = \{B, C, D\} \exp[i(\omega t - kz)], \quad (52)$$

where  $\omega$  is the angular frequency,  $k$  is the wave number,  $\bar{P}(r), \dots, \bar{W}(r)$  are unknown amplitude functions of the fluid, and  $B, C$ , and  $D$  are three constants standing for the complex wave amplitudes of the tube. Introducing equation (51) into equations (1) and (2) we obtain

$$-\frac{d\bar{P}}{dr} + \mu_v \left( \frac{d^2 \bar{U}}{dr^2} + \frac{1}{r} \frac{d\bar{U}}{dr} - \frac{\bar{U}}{r^2} - k^2 \bar{U} \right) - i\bar{\rho}\omega \bar{U} = 0, \quad (53)$$

$$\mu_v \left( \frac{d^2 \bar{V}}{dr^2} + \frac{1}{r} \frac{d\bar{V}}{dr} - \frac{\bar{V}}{r^2} - k^2 \bar{V} \right) - i\bar{\rho}\omega \bar{V} = 0, \quad (54)$$

$$ik\bar{P} + \mu_v \left( \frac{d^2 \bar{W}}{dr^2} + \frac{1}{r} \frac{d\bar{W}}{dr} - k^2 \bar{W} \right) - i\bar{\rho}\omega \bar{W} = 0, \quad (55)$$

$$\frac{d\bar{U}}{dr} + \frac{\bar{U}}{r} - ik\bar{W} = 0. \quad (56)$$

The solution of these ordinary differential equations is given as

$$\begin{aligned} \bar{P} &= -i\bar{\rho}\omega A_1 I_0(kr), \quad \bar{U} = k[A_1 I_1(kr) + A_2 J_1(sr)] \\ \bar{V} &= -ikA_3 J_1(sr), \quad \bar{W} = -i[kA_1 I_0(sr) + sA_2 J_0(sr)] \\ s^2 &= -k^2 - i\bar{\rho}\omega/\mu_v, \end{aligned} \quad (57)$$

where  $A_i$  ( $i = 1, 2, 3$ ) are three integration constants to be determined from the boundary conditions.

Before we attempt to solve the equations (47)–(49) it would be more convenient to calculate the stress components of the fluid body. Dropping the exponential factors their values on the midsurface may be given as follows

$$\begin{aligned} -\bar{\sigma}_{rr} &= \left[ -(i\bar{\rho}\omega + 2\mu_v k^2) I_0(k\bar{r}) + \frac{2\mu_v k}{\bar{r}} I_1(k\bar{r}) \right] A_1 \\ &\quad + \left[ \frac{2\mu_v k}{\bar{r}} J_1(s\bar{r}) - 2\mu_v k s J_0(s\bar{r}) \right] A_2 \\ -\bar{\sigma}_{r\theta} &= -i\mu_v k \left[ \frac{2}{\bar{r}} J_1(s\bar{r}) - s J_0(s\bar{r}) \right] A_3 \\ -\bar{\sigma}_{rz} &= 2i\mu_v k^2 I_1(k\bar{r}) A_1 + (2i\mu_v k^2 - \bar{\rho}\omega) J_1(s\bar{r}) A_2. \end{aligned} \quad (58)$$

Introducing equation (52) into equations (47)–(49) and utilizing equation (58) the following algebraic equations are obtained

$$\begin{aligned} &\left[ \rho h \omega^2 - N_z^0 k^2 + \frac{N_\theta^0 - h\alpha_{22}^0}{\bar{r}^2} \right] B \\ &\quad + \frac{ik}{\bar{r}} (2N_{\theta z}^0 + h\alpha_{23}^0) C + \frac{ikh\alpha_{21}^0}{\bar{r}} D \\ &\quad + \left[ -(i\bar{\rho}\omega + 2\mu_v k^2) I_0(k\bar{r}) + \frac{2\mu_v k}{\bar{r}} I_1(k\bar{r}) \right] A_1 \\ &\quad + \left[ \frac{2\mu_v k}{\bar{r}} J_1(s\bar{r}) - 2\mu_v k s J_0(s\bar{r}) \right] A_2 = 0 \end{aligned} \quad (59)$$

$$\begin{aligned}
& -\frac{ik}{\bar{r}}(2N_{\theta z}^0 + h\alpha_{32}^0)B \\
& + [\rho h\omega^2 - (N_z^0 + h\alpha_{33}^0)k^2]C - h\alpha_{31}^0 k^2 D \\
& - i\mu_v k \left[ \frac{2}{\bar{r}} J_1(s\bar{r}) - sJ_0(s\bar{r}) \right] A_3 = 0, \quad (60) \\
& -\frac{ik}{\bar{r}}(N_z^0 - N_{\theta}^0 + h\alpha_{12}^0)B - h\alpha_{13}^0 k^2 C \\
& + [\rho h\omega^2 - h\alpha_{11}^0 k^2]D + 2i\mu_v k^2 I_1(k\bar{r})A_1 \\
& + (2i\mu_v k^2 - \bar{\rho}\omega)J_1(s\bar{r})A_2 = 0. \quad (61)
\end{aligned}$$

The boundary conditions in equation (50) take the following form

$$i\omega B - k[I_1(k\bar{r})A_1 + J_1(s\bar{r})A_2] = 0 \quad (62)$$

$$\omega C + kJ_1(s\bar{r})A_3 = 0 \quad (63)$$

$$\omega D + kI_0(k\bar{r})A_1 + sJ_0(s\bar{r})A_2 = 0. \quad (64)$$

Eliminating the coefficients  $B$ ,  $C$ , and  $D$  between the equations (59)–(64) the following nondimensionalized homogeneous algebraic equations are obtained

$$\begin{aligned}
& \left\{ \left[ m \left( \Omega^2 - G\xi^2 + \frac{S - \alpha_{22}}{\bar{\lambda}_{\theta}^2} \right) + 2i\nu\lambda_z\Omega \right] \xi^2 f \right. \\
& \quad \left. + \left( \lambda_z\Omega^2 + \frac{\alpha_{21}m\xi^2}{\bar{\lambda}_{\theta}^2} - 2i\nu\lambda_z\Omega\xi^2 \right) \right\} A_1 \\
& + \left\{ \left[ m \left( \Omega^2 - G\xi^2 + \frac{S - \alpha_{22}}{\bar{\lambda}_{\theta}^2} \right) + 2i\nu\lambda_z\Omega \right] \xi\xi g \right. \\
& \quad \left. + \left( \frac{\alpha_{21}m}{\bar{\lambda}_{\theta}^2} - 2i\nu\lambda_z\Omega \right) \xi\xi \right\} A_2 \\
& + \left( \frac{m(2T + \alpha_{23})}{\bar{\lambda}_{\theta}} \right) \xi^2 \xi g A_3 = 0, \quad (65)
\end{aligned}$$

$$\begin{aligned}
& m\xi^2[(2T + \alpha_{32})f - \alpha_{31}]A_1 \\
& + m\xi\xi[(2T + \alpha_{32})g - \alpha_{31}]A_2 \\
& + \xi\{m[\Omega^2 - (G + \alpha_{33})\xi^2]\bar{\lambda}_{\theta}g \\
& + i\bar{\lambda}_{\theta}\lambda_z\nu\Omega(2g - 1)\}A_3 = 0, \quad (66)
\end{aligned}$$

$$\begin{aligned}
& \xi[m(G - S + \alpha_{12})\xi^2 f + m(\Omega^2 - \alpha_{11}\xi^2) \\
& \quad - 2i\nu\xi\bar{\lambda}_{\theta}^2\lambda_z\Omega f]A_1 \\
& + \xi[m(G - S + \alpha_{12})\xi^2 g + m(\Omega^2 - \alpha_{11}\xi^2) \\
& \quad + \bar{\lambda}_{\theta}^2\lambda_z(\Omega^2 - 2i\nu\Omega\xi^2)g]A_2 \\
& - m\bar{\lambda}_{\theta}\alpha_{13}\xi^3\xi g A_3 = 0, \quad (67)
\end{aligned}$$

where, for convenience, the following nondimensionalized quantities are introduced

$$\begin{aligned}
\mu &= \rho c_0^2, \quad \omega = \frac{c_0}{\bar{R}}\Omega, \quad N_z^0 = \mu hG, \quad N_{\theta}^0 = \mu hS \\
N_{\theta z}^0 &= \mu hT, \quad \alpha_{ij}^0 = \mu\alpha_{ij}, \quad k = \frac{\xi}{\bar{R}}, \quad s = \frac{\zeta}{\bar{R}} \\
\mu_v &= \bar{\rho}c_0\bar{R}\nu, \quad m = \frac{\rho H}{\bar{\rho}\bar{R}}, \quad \lambda_{\theta} = \frac{\bar{r}}{\bar{R}} \\
f &= \frac{I_1(\xi\bar{\lambda}_{\theta})}{\xi\bar{\lambda}_{\theta}I_0(\xi\bar{\lambda}_{\theta})}, \quad g = \frac{J_1(\zeta\bar{\lambda}_{\theta})}{\zeta\bar{\lambda}_{\theta}J_0(\zeta\bar{\lambda}_{\theta})}. \quad (68)
\end{aligned}$$

In order to have a nontrivial solution for the unknowns  $A_1$ ,  $A_2$ , and  $A_3$  the determinant of the coefficient matrix must vanish. If this operation is carried out the result will be too complicated and we shall not list it here. In what follows we shall study the long-wave limit.

### 3.1 Long-wave limit

Even for large arteries the wavelength is very large as compared to the midradius of arteries. Therefore the nondimensionalized wavelength  $\xi$  will be very small as compared to unity so that we may neglect the terms containing  $\xi$ . Moreover in this limiting case the function  $f$  approaches unity and  $\zeta$  approaches to  $i^{3/2}(\Omega/\nu)^{1/2}$ . In addition, as pointed out by Kuiken<sup>19</sup> and Bauer et al.,<sup>20</sup> the Womersley parameter  $\beta_0 = (\Omega/\nu)^{1/2}$  satisfies the condition  $|\xi|/\beta_0 \ll 1$ . Therefore we may neglect the terms with factors  $\nu\Omega$  and  $\nu^2\Omega^2$  appearing in the expansion. Under these simplifying assumptions the dispersion relation reduces to

$$B_1c^6 + B_2c^4 + B_3c^2 + B_4 = 0, \quad (69)$$

where the coefficients  $B_i$  ( $i = 1, 2, 3, 4$ ) are defined by

$$\begin{aligned}
B_1 &= \lambda_{\theta}^2\lambda_z(m + g\lambda_{\theta}^2\lambda_z) \\
B_2 &= m^2\left(g - \frac{1}{2}\right)(\alpha_{22} - S) \\
& \quad + m\lambda_{\theta}^2\lambda_z\left[g\left(\alpha_{12} + \alpha_{21} + G - \frac{S}{2} - \frac{\alpha_{22}}{2}\right) - \alpha_{11}\right] \\
& \quad - \lambda_{\theta}^2\lambda_z(G + \alpha_{33})(m + g\lambda_{\theta}^2\lambda_z) \\
B_3 &= -(G + \alpha_{33})\left[m^2\left(\frac{1}{2} - g\right)(S - \alpha_{22}) \right. \\
& \quad \left. + m\lambda_{\theta}^2\lambda_z\left[\frac{1}{2} - g(S - \alpha_{22}) \right. \right. \\
& \quad \left. \left. + g(G - S + \alpha_{12} + \alpha_{21}) - \alpha_{11}\right] \right. \\
& \quad \left. + m^2\left(\frac{1}{2} - g\right)[(S - G)\alpha_{21} \right. \\
& \quad \left. - S\alpha_{11} + \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}] \right]
\end{aligned}$$

$$\begin{aligned}
 & + \lambda_\theta m \left[ T(2T + \alpha_{23} + \alpha_{32}) \right. \\
 & \times (2mg - \lambda_\theta^2 \lambda_z g - m) + 2\lambda_\theta^2 \lambda_z g T(\alpha_{31} + \alpha_{13}) \\
 & + g\lambda_\theta^2 \lambda_z (\alpha_{32} \alpha_{13} + \alpha_{23} \alpha_{31} - \alpha_{31} \alpha_{13}) \\
 & \left. + \alpha_{23} \alpha_{32} \left( mg - \frac{\lambda_\theta^2 \lambda_z}{2} - \frac{m}{2} \right) \right], \\
 B_4 = & m^2 \left( g - \frac{1}{2} \right) \{ (G + \alpha_{33})(S\alpha_{21} - G\alpha_{21} - S\alpha_{11} \\
 & + \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}) \\
 & + [2T(\alpha_{12} \alpha_{31} + \alpha_{21} \alpha_{13} + G\alpha_{31} \\
 & - \alpha_{32} \alpha_{11} - \alpha_{23} \alpha_{11} - S - 2T\alpha_{11}) \\
 & + S(\alpha_{13} \alpha_{31} - \alpha_{23} \alpha_{31}) + G\alpha_{23} \alpha_{31} \\
 & + (\alpha_{23} \alpha_{31} \alpha_{12} + \alpha_{32} \alpha_{21} \alpha_{13} \\
 & - \alpha_{23} \alpha_{32} \alpha_{11} - \alpha_{13} \alpha_{31} \alpha_{22}) \}. \quad (70)
 \end{aligned}$$

If the initial twist is zero, i.e.,  $K \equiv 0$ , the dispersion relation in equation (69) reduces to

$$(D_0 c^4 + D_1 c^2 + D_2)[c^2 - (G + \alpha_{33})] = 0 \quad (71)$$

where the coefficients  $D_i$  ( $i = 0, 1, 2$ ) are defined by

$$\begin{aligned}
 D_0 = & \lambda_\theta^2 \lambda_z (m + g\lambda_\theta^2 \lambda_z) \\
 D_1 = & m^2 \left( g - \frac{1}{2} \right) (\alpha_{22} - A) \\
 & + m\lambda_\theta^2 \lambda_z \left[ g \left( G - \frac{S}{2} - \frac{\alpha_{22}}{2} + \alpha_{12} + \alpha_{21} \right) - \alpha_{11} \right] \\
 D_2 = & m^2 \left( g - \frac{1}{2} \right) [\alpha_{21}(G - S) + \alpha_{11}S \\
 & + \alpha_{12} \alpha_{21} - \alpha_{11} \alpha_{22}]. \quad (72)
 \end{aligned}$$

As is seen from equation (71), in this special case, the torsional wave is not coupled with the remaining waves. The existence of an initial twist provides the coupling between these three waves.

In general it is quite difficult to investigate the dispersion relation in equation (69) by analytical means. Therefore in what follows we shall study the dispersion equation by numerical means.

#### 4. Numerical analysis and discussion

In this section we shall study the dispersion equation by numerical means. For that purpose we decompose the complex phase velocity,  $c = \Omega/\xi$ , into its real and imaginary parts as

$$c = X + iY. \quad (73)$$

Then the speed of propagation  $v$  and the transmission

coefficient  $\chi$  may be expressed as

$$v = \frac{X^2 + Y^2}{X}, \quad \chi = \exp \left[ -2\pi \frac{Y}{X} \right]. \quad (74)$$

We further need the expressions of nondimensionalized coefficients  $\alpha_{ij}$  and the initial membrane forces, which may be given by,

$$\begin{aligned}
 \alpha_{11} = & \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} + \lambda_z^2 \right) + 2\gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right)^2 \right] F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{12} = & \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} - \lambda_z^2 \right) + 2\gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right. \\
 & \left. \times \left( \bar{\lambda}_\theta^2 + \varepsilon^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{13} = & 2\lambda_z \varepsilon \left[ 1 + \gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{21} = & \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} - \lambda_\theta^2 - \varepsilon^2 \right) + 2\gamma \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right. \\
 & \left. \times \left( \bar{\lambda}_\theta^2 + \varepsilon^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{22} = & \left[ \left( \frac{3}{\bar{\lambda}_\theta^2 \lambda_z^2} + \lambda_\theta^2 + \varepsilon^2 \right) + 2\gamma \left( \lambda_\theta^2 + \varepsilon^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right)^2 \right] \\
 & \times F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{23} = & 2\lambda_z \varepsilon \left[ 1 + \gamma \left( \bar{\lambda}_\theta^2 + \varepsilon^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) \right] F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{31} = & 2\gamma \lambda_z \varepsilon \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{32} = & 2\gamma \lambda_z \varepsilon \left( \lambda_\theta^2 + \varepsilon^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) F(\bar{\lambda}_\theta, \lambda_z) \\
 \alpha_{33} = & \left( \lambda_\theta^2 + \varepsilon^2 + \lambda_z - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} + 2\gamma \varepsilon^2 \right) F(\bar{\lambda}_\theta, \lambda_z), \quad (75)
 \end{aligned}$$

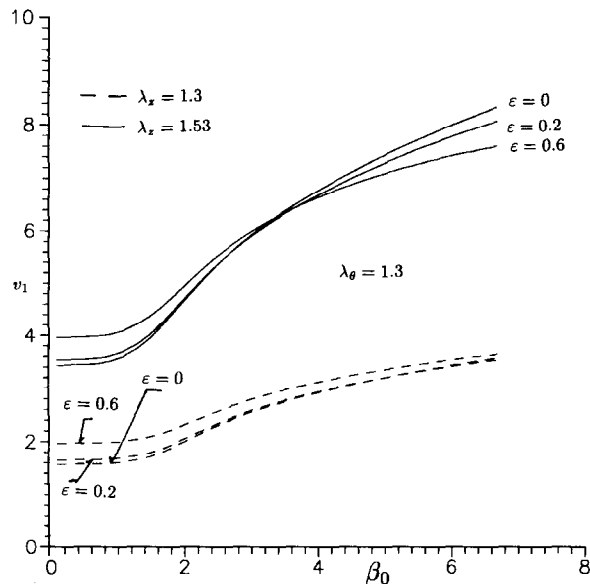
and

$$\begin{aligned}
 S = & \left( \bar{\lambda}_\theta^2 + \varepsilon^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) F(\bar{\lambda}_\theta, \lambda_z) \\
 G = & \left( \lambda_z^2 - \frac{1}{\bar{\lambda}_\theta^2 \lambda_z^2} \right) F(\bar{\lambda}_\theta, \lambda_z), \quad T = \lambda_z \varepsilon F(\bar{\lambda}_\theta, \lambda_z), \quad (76)
 \end{aligned}$$

where  $\varepsilon = K\bar{r}$  stands for the twist parameter.

For numerical purposes one also needs the value of the material constant  $\gamma$ . Employing the experimental results

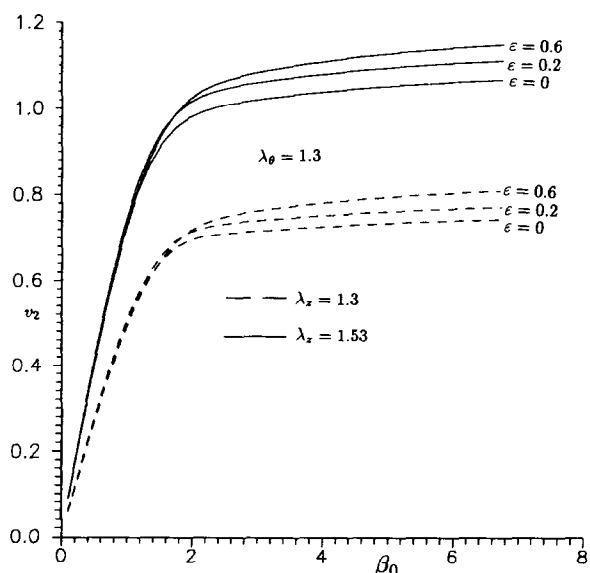




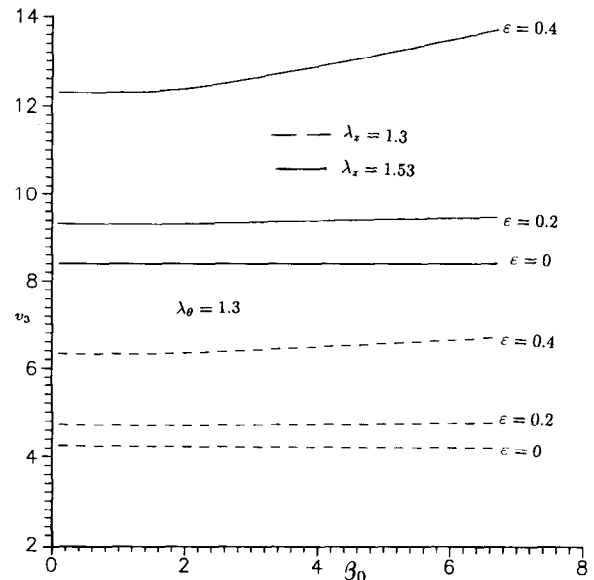
**Figure 2.** Variation of primary wave speed with the Womersley parameter, stretch ratios, and twist angle.

by Simon et al.<sup>21</sup> on the canine abdominal artery with geometrical characteristics  $H = 0.06$  cm,  $\bar{R} = 0.35$  cm we have determined (Demiray<sup>22</sup>) the material coefficient  $\gamma$  as  $\gamma = 1.948$ . We further notice that the mass densities of the arterial wall and the blood are quite close, i.e.,  $\rho/\bar{\rho} \approx 1$  and  $m \approx H/\bar{R}$ . The dispersion relation is evaluated numerically for various stretch ratios,  $\epsilon$  and Womersley parameter  $\beta_0 = (\Omega/\nu)^{1/2}$ , and the results are depicted in Figures 2–7.

The variation of primary wave speed with the Womersley parameter, twist angle, and stretch ratios is given in Figure 2. Examination of this figure reveals that the wave speed increases with the Womersley parameter, axial

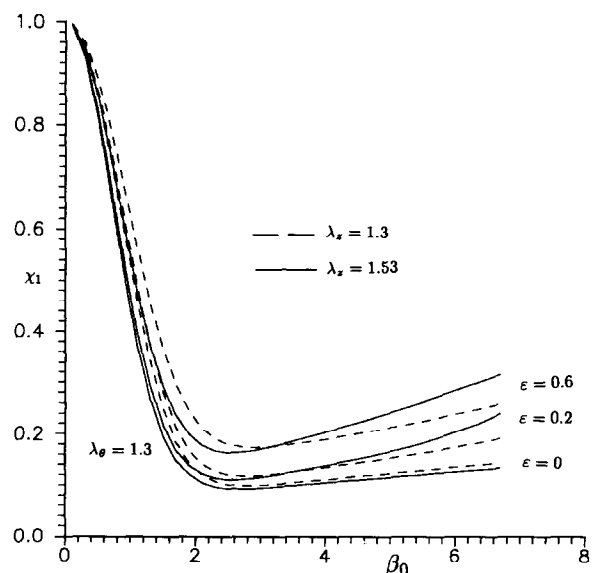


**Figure 3.** Variation of secondary wave speed with the Womersley parameter, stretch ratios, and twist angle.

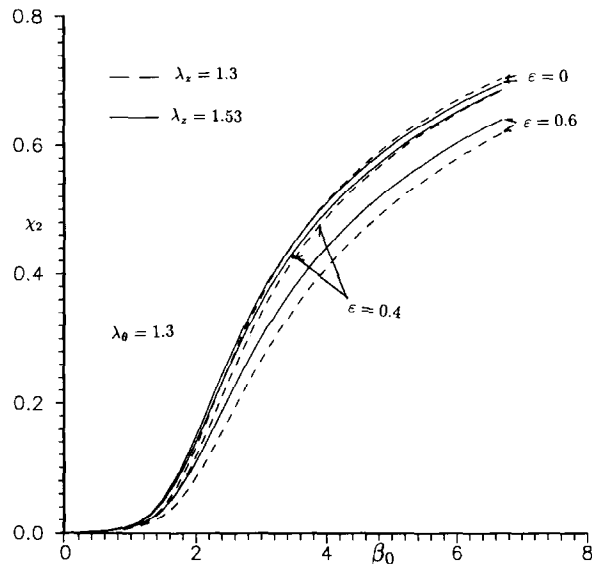


**Figure 4.** Variation of torsional wave speed with the Womersley parameter, stretch ratios, and twist angle.

stretch, and twist angle for low frequencies. However, for large values of the frequency the wave speed starts to decrease with the twist parameter. Nevertheless this wave speed is not very sensitive to changes in initial twist angle. The variation of secondary wave speed with the same set of variables is shown in Figure 3. This speed increases very fast with the Womersley parameter and suddenly slows down. This wave speed also increases with increasing axial stretch ratio and twist angle, but it is not so sensitive to changes in initial twist angle. Finally the variation of torsional wave speed with the same set of parameters is shown in Figure 4. As is seen for a vanishing twist angle this wave is not dispersive but the existence of an initial



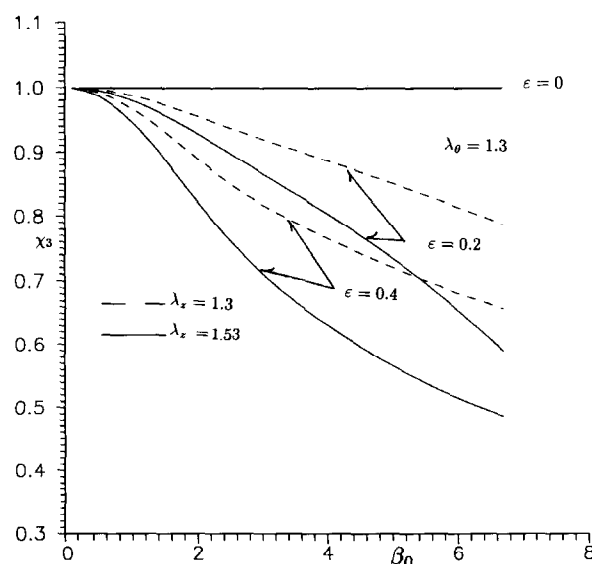
**Figure 5.** Variation of transmission coefficient of primary with Womersley parameter stretch ratios and twist angle.



**Figure 6.** Variation of transmission coefficient of secondary wave with the Womersley parameter, stretch ratios, and twist angle.

twist leads to a dispersive wave. This wave speed also increases with the above-mentioned set of parameters. Contrary to previous waves this speed is quite sensitive to changes in twist angle. Here we should note that when the initial twist angle  $\epsilon = 0$  the first two waves reduce to the case studied by Rachev,<sup>9</sup> whereas the last one reduces to the case we studied in Ref. 12.

The variations of transmission coefficients with stretch ratio, twist angle, and Womersley parameter are depicted in Figures 5–7. Figure 5 shows that the transmission



**Figure 7.** Variation of transmission coefficient of torsional wave with the Womersley parameter, stretch ratios, and twist angle.

coefficient of the primary wave first decreases very fast with the Womersley parameter then starts to increase gradually, but it increases with twist angle. The variation of transmission coefficient of the secondary wave is shown in Figure 6. This coefficient increases with the Womersley parameter and axial stretch but decreases with twist angle. Finally the variation of the transmission coefficient of the torsional wave is given in Figure 7. This coefficient is very sensitive to the changes in twist angle. For zero twist angle the totality of wave is transmitted. However the existence of an initial twist causes the wave amplitude to disperse and diminish with distance. This coefficient decreases with the Womersley parameter stretch ratio and the twist angle.

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